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MINIMAL IMMERSIONS OF COMPACT IRREDUCIBLE HOMOGENEOUS RIEMANNIAN MANIFOLDS

PETER LI

0. Introduction

The purpose of this paper is to study the space of isometric minimal immersions of a compact irreducible homogeneous Riemannian manifold M^m into a standard sphere S^n. By a theorem of Takahashi [6], any compact irreducible homogeneous Riemannian manifold can be isometrically minimally immersed into some $S^n(r)$ using its spaces of eigenfunctions satisfying the equation

$$\Delta\varphi = -\lambda\varphi \tag{0.1}$$

for some constant λ. The set λ such that (0.1) has nontrivial solution is called the spectrum of the Laplace operator Δ on M, denoted by Spec(M). It is also known that [4] the coordinate functions of any isometric minimal immersions of M into $S^n \subseteq \mathbf{R}^{n+1}$ are eigenfunctions of the Laplacian. In 1971, do Carmo and Wallach [2] consider the case when M is also a standard sphere. However, some of their results also hold when M is a compact irreducible homogeneous Riemannian manifold.

The main result which we have obtained in the paper is a classification theorem of all isometric minimal immersions. In fact, we show that if $\Phi: M \to S^n(r)$ is an isometric minimal immersion, then $\Phi(M) = N$ is also a compact irreducible homogeneous Riemannian manifold which is embedded in $S^n(r)$. The map $\Phi: M \to N$ is in fact a covering map, and N inherits the homogeneous structure of M.

As an application of the above theorem, we show that if N is a compact Riemannian manifold which is isometrically covered by M. Then N can be isometrically minimally immersed into some $S^n(r)$ iff N has the induced homogeneous structure of M. We also give necessary and sufficient conditions for an eigenspace E_λ of M to be invariant under the group of deck transformations $\Gamma(N)$ with respect to the covering map $\pi: M \to N$. An

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interesting corollary of this is that if N is a lens space which is k-fold covered by S^{2m-1}, then N cannot be isometrically minimally immersed into any standard spheres unless $k = 1$ or 2.

In the last section, we consider the question whether a compact irreducible homogeneous Riemannian manifold can always be isometrically minimally embedded into some S^n. Using the Weyl formula, we show that if $M = G/H$, where G acts effectively on M, and if the center $Z(G)$ of G is a cyclic group, then there exists infinitely many eigenspaces of M which give isometric minimal embeddings of M into $S^n(r)$.

We will adopt the convention that any isometric minimal immersioin $\Phi: M \to S^n(r)$ is full, i.e., $\Phi(M)$ is not contained in any totally geodesic $S^p(r)$ of $S^n(r)$ with $p < r$.

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1. Spaces of isometric minimal immersions

Definition. A homogeneous manifold $M^m = G/H$ is said to be irreducible if its isometry group G is compact and its isotropy subgroup H acts irreducibly on the tangent space at $eH \in M$, where e is the identity element of G. In addition, we also assume that G acts effectively on M.

For the sake of completeness, we will outline the proof of do Carmo and Wallach for general compact irreducible homogeneous Riemannian manifolds.

Proposition 1. *Let $\Phi: M^m \to S^n(r)$ be an isometric minimal immersion of M into $S^n(r)$. Then $r^2 = m/\lambda$ for some $\lambda \in \mathrm{Spec}(M)$. Moreover, for a fixed λ, the set of such isometric minimal immersions can be parametrized by a compact convex body in a finite dimensional vector space.*

Proof. If we consider $S^n(r) \subseteq \mathbf{R}^{n+1}$, then it is known that [4] the coordinate functions of $\Phi: M^m \to \mathbf{R}^{n+1}$ are eigenfunctions with eigenvalue m/r^2. Up to orthogonal transformation, we may assume that $\Phi = A\Psi$, where A is a semi-positive symmetric matrix and Ψ denotes the standard immersion given by $\Psi = (\alpha\varphi_1, \cdots, \alpha\varphi_{k+1})$ with $\{\varphi\}_{i=1}^{k+1}$ being an orthonormal basis of $E_\lambda = \{f | \Delta f = -\lambda f\}$, $\lambda = m/r^2$.

Let us denote $V_1 = d\Psi(T_x M) \subseteq T_{\Psi(x)} S^k(r)$ and $S^2(V_1) = \{$symmetric squares of $V_1\}$. Also let $W_0 = \{G \cdot S^2(V_1)\}_{\mathbf{R}}$-linear span of the orbit of $S^2(V_1)$ in $S^2(E_\lambda)$ where E_λ is identified to $T_{\Psi(x)}\mathbf{R}^{k+1}$.

One can identify the symmetric square $S^2(E_\lambda)$ with the space of symmetric linear maps of E_λ, where the linear map is defined by

$$uv(t) = \frac{1}{2}(\langle u, t\rangle v + \langle v, t\rangle u) \tag{1.1}$$

for $t \in E_\lambda$ and $uv \in S^2(E_\lambda)$. One obtains an induced inner product on $S^2(E_\lambda)$ given by $(A, B) = \mathrm{tr}(AB)$, for all $A, B \in S^2(E_\lambda)$, and the induced action of $g \in G$ on $S^2(E_\lambda)$ is given by $g \cdot A = gAg^{-1}$. Clearly $\langle Au, Av\rangle = (A, uv)$. We define $W = \{u \in S^2(E_\lambda) | (u, W_0) = 0\}$.

Now we claim that $\Phi = A\Psi$ is an isometric immersion iff $A^2 - I \in L$, where $L = \{c \in W | C + I \geqslant 0\}$. In fact, $A\Psi$ is an isometric immersion iff

$$\langle AgX^*, AgX^*\rangle = 1 \tag{1.2}$$

for all $g \in G, X \in S_xM, X^* = d\Psi(X)$. However, this is equivalent to

$$(A^2 - I, g \cdot (X^*)^2) = 0, \tag{1.3}$$

which means $A^2 - I \in W$, hence $A^2 - I \in L$ since $A \geqslant 0$ and is symmetric. The converse follows similarly. Of course, by Takahashi's theorem, if Φ is an isometric immersion then Φ is minimal.

Therefore the equivalent classes of isometric minimal immersions can be parametrized by the set $L \subset W$. Clearly L is a convex set with boundary. Moreover since $\mathrm{tr}\, A^2 = \dim E_\lambda$ for $A^2 - I \in L$, we conclude that if $c \in L$, then $\mathrm{tr}\, c = 0$. This implies that the eigenvalues of the elements in L are bounded, hence L is compact. In fact, the boundary points of L correspond to A being singular, i.e., $n < \dim E_\lambda - 1$.

2. A classification theorem

Definition. A function $f_0 \in E_\lambda$ is said to be the normalized zonal function at $x_0 \in M$ with respect to E_λ if it satisfies the following properties:

(i) f_0 is constant on the orbit of H_0 = isotropy subgroup of G which fixes x_0,

(ii) f_0 is perpendicular (in the L^2 sense) to the set of functions in E_λ which vanish at x_0,

(iii) $f_0(x_0) = \|f_0\|_\infty$,

(iv) $\|f_0\|_2 = 1$.

Proposition 2. *In each eigenspace E_λ of M and for a fixed $x_0 \in M$, there exists a unique normalized zonal function at x_0 with respect to E_λ.*

Proof. The proof of this proposition is contained in [3] and [5]. However, we will sketch the proof here.

Let us consider the space $E = \{f \in E_\lambda | \langle f, g\rangle = 0$ for all g such that $g(x_0) = 0\}$. It is easy to see that E is a 1-dimensional subspace of E_λ. Consider $f_0 \in E$ such that $\|f_0\|_2 = 1$, and $f_0(x_0) \neq 0$. Since E is invariant under the action of H_0, f_0 satisfies conditions (i), (ii) and (iv).

On the other hand, if we define the function

$$F(x) = \sum_{i=1}^{k+1} \varphi_i^2(x), \quad \text{for } x \in M, \tag{2.1}$$

where $\{\varphi_i\}_{i=1}^{k+1}$ is an orthonormal basis of E_λ, by the homogeneity assumption and the fact that $F(x)$ is well defined under an orthogonal change of basis of E_λ, $F(x) =$ constant. In particular,

$$F(x_0) = F(x). \tag{2.2}$$

If we pick an orthonormal basis such that $f_0 = \varphi_1$, then

$$F(x_0) = f_0^2(x_0). \tag{2.3}$$

Hence

$$\sum_{i=1}^{k+1} \varphi_i^2(x) = f_0^2(x_0). \tag{2.4}$$

Integrating both sides yields

$$k + 1 = V \cdot f_0^2(x_0), \tag{2.5}$$

where $V = V(M)$ is the volume of M. But

$$\frac{k+1}{V} = \sum_{i=1}^{k} \varphi_i^2(x)$$

implies that

$$\|\varphi\|_\infty^2 \leqslant \frac{k+1}{V}, \quad \text{for all } \varphi \in E_\lambda. \tag{2.6}$$

In particular,

$$\|f_0\|_\infty^2 \leqslant \frac{k+1}{V} = f_0^2(x_0),$$

which proves the proposition.

Lemma 3. *Let Φ: $M \to S^n(r)$ be an isometric minimal immersion. Suppose Φ corresponds to an interior point of L as discussed in Proposition* 1. *If N denotes the image of Φ in $S^n(r)$, then N is an isometrically minimally embedded submanifold of $S^n(r)$. Moreover Φ: $M \to N$ is a covering map.*

Proof. Clearly, we need only to show that the preimage set of each point $z \in N$ consists of exactly q points. By scaling, we may assume that

$$\dim E_\lambda = V(M). \tag{2.7}$$

By an orthonormal change of basis, if necessary, we may assume N contains p = north pole of $S^n(r)$. We claim that if $\Phi(x_0) = p$ then the preimage $\Phi^{-1}(p)$ of p consists of points in M which take on the maximum value of the normalized zonal function f_0 at x_0.

Indeed, if $\Phi(x) = (\varphi_1(x), \cdots, \varphi_{n+1}(x))$, then $\Phi(x_0) = p$ implies $\varphi_1(x_0) = r$ and $\varphi_\alpha(x_0) = 0$ for $\alpha \neq 1$. This means that $\varphi_\alpha \in E_0 = \{f \in E_\lambda | f(x_0) = 0\}$. Since by assumption $n + 1 = \dim E_\lambda = k + 1$, we conclude that $\langle \varphi_\alpha \rangle_{\alpha=2}^{n+1} = E_0$. Hence $\varphi_1 = af_0 + bg$ for some $a, b \in \mathbf{R}$ and $g \in E_0$. However, by $r = \varphi_1(x_0) = af_0(x_0)$, (2.5) and (2.7) we have

$$r = af_0(x_0) = a. \tag{2.8}$$

Hence

$$\varphi_1 = rf_0 + bg. \tag{2.9}$$

If $x \in \{\text{maximal points of } f_0\}$, then $f_0(x) = 1$. From (2.6) we conclude that

$$g(x) = 0, \quad \text{for } g \in E_0, \tag{2.10}$$

which means $E_0 = E_1 = \{f \in E_\lambda | f(x) = 0\}$ because $\dim E_0 = n = \dim E_1$. Therefore

$$\varphi_1(x) = rf_0(x) = r$$

and

$$\varphi_\alpha(x) = 0, \quad \alpha \neq 1,$$

which implies $\Phi(x) = p$.

Conversely, if $\Phi(x) = p$, then $\varphi_1(x) = r$ and $\varphi_\alpha(x) = 0$ for $\alpha \neq 1$. Thus

$$\varphi_\alpha \in E_1 = \{f \in E_\lambda | f(x) = 0\},$$

and $E_1 = E_0$. It follows that

$$r = \varphi_1(x) = rf_0(x) + bg(x) = rf_0(x). \tag{2.11}$$

However $f_0(x) = 1$ implies that x takes on the maximum value of f_0. The lemma then follows directly.

Theorem 4. *Let Φ: $M \to S^n(r)$ be an isometric minimal immersion of M into $S^n(r)$. Then the image N of Φ is a compact homogeneous space which is isometrically minimally embedded in $S^n(r)$. Moreover, the homogeneous structure of N is the one induced from M, i.e., the group of deck transformations $\Gamma(N)$ with respect to the covering map Φ: $M \to N$ is contained in the center $Z(G)$ of G.*

Proof. We will first prove the theorem for those Φ which correspond to the interior points of L. We claim that for any $g \in G$, g commutes with the element of $\Gamma(N)$.

Observe that g preserves fibers over N. Indeed if $\bar{x}, \bar{y} \in N$, then $\Phi^{-1}(\bar{x})$ and $\Phi^{-1}(\bar{y})$ coincide with the sets $\{x \in M | f_1(x) = \|f_1\|_\infty\}$ and $\{x \in M | f_2(x) = \|f_2\|_\infty\}$ respectively, where f_1 and f_2 are normalized zonal functions at preimage points of $\bar{x}$ and $\bar{y}$. However if $g \in G$ and $g(x_1) = y_1$ with $x_1 \in \Phi^{-1}(\bar{x})$ and $y_1 \in \Phi^{-1}(\bar{y})$, then $g \cdot f_2 = f_2 \circ g$ is a zonal function at y_1. Hence by uniqueness $f_1 = g \cdot f_2$. This shows $\Phi^{-1}(\bar{x}) = \Phi^{-1}(\bar{y})$.

Since G is a Lie group, in order to show the claim, it suffices to show that g commutes with $\Gamma(N)$ for those g which send x to nearby points. Let U be a sufficiently small neighborhood of $\bar{x} \in N$ such that U is evenly covered by disjoint neighborhoods $\{U_i\}_{i=1}^q$ of $\{x_i\}_{i=1}^q = \Phi^{-1}(\bar{x})$, with $x_i \in U_i$ for all $1 \leqslant i \leqslant q$. We would like to show that g commutes with $\Gamma(N)$ if $g(x_1) \in U_1$. Let $\bar{y} = \Phi(g(x_1))$ and $\{y_i\}_{i=1}^q = \Phi^{-1}(\bar{y})$ such that $y_i \in U_i$. Clearly we need only to show that $g(x_i) = y_i$. By picking U sufficiently small and using the fact that g is an isometry, we have $g(x_i) \in U_i$. However g preserving fibers implies that $y_i = g(x_i)$ because $\{U_i\}$ are disjoint. This proves the theorem for those Φ which are the interior points of L. For the boundary points we can utilize a continuation argument. In fact, if we take a path through the interior of L to a boundary point Φ, then it is clear that by continuity the theorem also holds for the boundary points.

Remark. Any set of eigenfunctions from an eigenspace E_λ gives an isometric minimal immersion of M into $S^n(r)$ with $r^2 = m/\lambda$ iff they satisfy the algebraic criterion described in §1.

3. Applications

In case when M is also a standard sphere S^m of radius 1, Theorem 4 yields the following.

Theorem 5. *If $\Phi: S^m \to S^n(r)$ is an isometric minimal immersion, then $r^2 = m/\lambda$ for some $\lambda \in$ Spec(S^m). Moreover $\Phi(S^m)$ is either an embedded sphere or an embedded projective space. In fact, if* Spec$(S^m) = \{0 = \lambda_0 < \lambda_1 < \lambda_2 \cdots\}$ *(multiplicity not included), then Φ corresponds to embeddings of S^m if $r^2 = m/\lambda_{2i+1}$ for $0 \leqslant i < \infty$, and it corresponds to embedding of* $\mathbf{RP}^m$ *if $r^2 = m/\lambda_{2i}$ for $1 \leqslant i < \infty$.*

Proof. By Theorem 4, $\Phi(S^m)$ is an embedded homogeneous space covered by S^m with the induced homogeneous structure. This implies that the set of preimages of a point $z \in \Phi(S^m)$ is contained in the fixed point set of the isotropy subgroup of $x_0 \in \Phi^{-1}(z)$. Since the isotropy subgroup H_0 of $x_0 \in S^m$ has orbits homeomorphic to S^{m-1} with the exception of x_0 and its antipodal point, this means that $\Phi: S^m \to \Phi(S^m)$ is at most a 2-fold covering. Hence

$\Phi(S^m)$ is either S^m or $\mathbf{RP}^m$. However, it is known [1] that the eigenfunctions of S^m with eigenvalue λ_i are spanned by the harmonic homogeneous polynomials on $\mathbf{R}^{m+1}$ of degree i. Hence $-f(x) = f(-x)$ for $f \in E_{\lambda_{2i+1}}$ for $0 \leqslant i < \infty$, and $f(x) = f(-x)$ for $f \in E_{\lambda_{2i}}$ for $1 \leqslant i < \infty$. This proves the theorem.

Corollary 6. *Suppose N is a lens space which is isometrically k-fold covered by S^{2m-1}. Then N cannot be isometrically minimally immersed into any standard spheres if $k > 2$.*

Proof. Suppose on the contrary that Φ: N $\to S^n(r)$ is an isometric minimal immersion. Let π: $S^{2m-1} \to N$ be the covering map. Consider the composition $\Phi \circ \pi$: $S^{2m-1} \to S^n(r)$ which is clearly an isometric minimal immersion of S^{2m-1}. Moreover, the image $\Phi \circ \pi(S^{2m-1}) = \Phi(N)$ is at least k-fold covered by S^{2m-1}. But this contradicts Theorem 5 if $k > 2$.

Remark. In fact, the proof of Corollary 6 shows that if π: $M \to N$ is a covering map, then N can be isometrically immersed into some $S^n(r)$ iff N has the induced homogeneous structure of M.

In the general setting of an isometric covering π: $M \to N$, where M and N are only compact Riemannian manifolds, it is obvious that the eigenfunctions of N can be lifted to be eigenfunctions of M. If $\lambda \in \operatorname{Spec}(N)$, we denote the eigenspaces of N and M with eigenvalue λ by $\bar{E}_\lambda$ and E_λ respectively. Let $\pi^*(\bar{E}_\lambda)$ be the pulled back of $\bar{E}_\lambda$ to M, then $\pi^*(\bar{E}_\lambda) \subseteq E_\lambda$. It is natural to ask the following question: When does $\pi^*(\bar{E}_\lambda) = E_\lambda$? For the case where M is an irreducible homogeneous space, this question can be completely answered.

Theorem 7. *Let π: $M \to N$ be an isometric covering map. Then $\pi^*(\bar{E}_\lambda) = E_\lambda$ for all $\lambda \in \operatorname{Spec}(N)$ iff N inherits the homogeneous structure from M, i.e., $\Gamma(N) \subseteq Z(G)$.*

Proof. First we show that if there exists $\lambda \in \operatorname{Spec}(N)$ such that $\pi^*(\bar{E}_\lambda) = E_\lambda$, then $\Gamma(N) \subseteq Z(G)$. Let Φ: $M \to S^n(r)$ be the standard immersion by an orthonormal basis of E_λ. However $E_\lambda = \pi^*(\bar{E}_\lambda)$ means that the eigenfunctions are invariant under $\Gamma(N)$. Theorem 4 then implies that there exists $\tilde{N}$ which is covered by M and $\Gamma(\tilde{N}) \subseteq Z(G)$. Moreover $\tilde{N}$ is the embedded image of Φ. On the other hand, since Φ is invariant under $\Gamma(N)$ we have the following diagram

$$M \xrightarrow{\pi} N \xrightarrow{\theta} \tilde{N}$$

with $\theta \circ \pi = \tilde{\pi}$ and $\Gamma(\tilde{N}) \supseteq \Gamma(N)$. However $\Gamma(\tilde{N}) \subseteq Z(G)$, hence $\Gamma(N) \subseteq Z(G)$.

Conversely, suppose $Z(G) \supseteq \Gamma(N)$. Then N is also an irreducible homogeneous manifold. Therefore for any $\lambda \in \operatorname{Spec}(N)$, E_λ gives an isometric minimal immersion Φ: $N \to S^n(r)$ where $r^2 = m/\lambda$. This means that

$\Phi \circ \pi: M \to S^n(r)$ is an isometric minimal immersion of M into $S^n(r)$. By Theorem 4, we have

$$M \xrightarrow{\pi} N \xrightarrow{\Phi} \Phi(N) = \tilde{N}$$

where $\Gamma(\tilde{N}) \subseteq Z(G)$. However the proof of Theorem 4 implies that the image of the standard isometric minimal immersion of M into $S^k(r)$ by an orthonormal basis of E_λ is isometric to $\tilde{N}$. This implies that the eigenfunctions in E_λ are $\tilde{\Gamma}$-invariant, hence also Γ-invariant. This completes the proof of Theorem 7.

Remark. Theorem 7 actually shows that if $E_\lambda = \pi^*(\bar{E}_\lambda)$ for some $\lambda \in \mathrm{Spec}(N)$ then $E_\lambda = \pi^*(\bar{E}_\lambda)$ for all $\lambda \in \mathrm{Spec}(N)$.

When $M = S^{2m-1}$ and N a lens space k-fold covered by M. Then $E_\lambda \neq \pi^*(\bar{E}_\lambda)$ for all $\lambda \in \mathrm{Spec}(N)$ iff $k > 2$.

4. Embeddings

The above discussion gave us a rather clear picture of isometric minimal immersions of a compact irreducible homogeneous Riemannian manifold into a standard sphere. It is natural to ask if such a manifold M can always be isometrically minimally embedded into a standard sphere. By Theorem 4, this is equivalent to asking if there exists an eigenfunction on M which is not invariant under any subgroup of $Z(G)$. The next theorem gives conditions which guarantee the existence of infinitely such eigenfunctions

Theorem 8. *If $Z(G)$ is a cyclic group, then there exist infinitely many eigenfunctions which are not invariant under any subgroup of $Z(G)$.*

Since each eigenspace E_λ of M are of finite dimensions, we conclude

Corollary 9. *If $Z(G)$ is a cyclic group, then there exist infinitely many eigenspaces E_λ of M which give isometric minimal embeddings of M into $S^n(r)$.*

Before we prove Theorem 8, let us point out some elementary properties of $Z(G)$.

Lemma 10. *$Z(G)$ is a finite group, and $Z(G) \cap H = \{e\}$.*

Proof. Let x_0 be any point in M, and denote the orbit of x_0 under $Z(G)$ by $Z(x_0)$. Clearly $Z(x_0)$ is contained in the fixed point set of H_0. Indeed, if $h \in H_0$ and $z \in Z(G)$, then

$$hz(x_0) = zh(x_0) = z(x_0). \tag{4.1}$$

Hence if $Z(G)$ is not finite, by compactness there exist $z, z' \in Z(G)$ which are sufficiently close to each other. Let $x_0 \in M$ be the point which represents the coset zH. Then $z'(x_0)$ will be sufficiently close to x_0. If γ is the unique minimizing geodesic joining x_0 and $z'(x_0)$, then γ is invariant under H_0, since

x_0 and $z'(x_0)$ are invariant and γ is unique. However this implies the vector tangent to γ at x_0 is invariant under H_0, which contradicts the irreducibility assumption of H.

To prove that $Z(G) \cap H = \{e\}$, it suffices to show that if $z \in Z(G)$ where $z \neq e$, then z has no fixed point. Assume $x \in M$ is a fixed point of z. By the effectiveness of G, there exist points y_1 and y_2 in M such that $z(y_1) = y_2$. Let $g \in G$ be the isometry which sends y_2 to x. Now consider

$$z^{-1}gz(y_1) = z^{-1}g(y_2) = z^{-1}(x) = x. \tag{4.2}$$

On the other hand, since $z \in Z(G)$,

$$z^{-1}gz(y_1) = g(y_1), \tag{4.3}$$

which implies $g(y_1) = x$. However $g(y_2) = x$ and $y_1 \neq y_2$, which is a contradiction. Thus the proof is complete.

In general, let Z be a finite abelian group, and $S = \{K_\alpha\}_{\alpha=1}^{q}$ be the set of proper subgroups of Z. We denote $K_{\alpha_1 \cdots \alpha_p}$ to be the subgroup generated by $\bigcup_{i=1}^{p} K_{\alpha_i}$.

Proposition 11. *The equation*

$$|Z| = \sum_{\alpha} \frac{|Z|}{|K\alpha|} - \sum_{\alpha_1<\alpha_2} \frac{|Z|}{|K_{\alpha_1\alpha_2}|} + \sum_{\alpha_1<\alpha_2<\alpha_3} \frac{|Z|}{|K_{\alpha_1\alpha_2\alpha_3}|} - \cdots \pm \frac{|Z|}{|K_{12\cdots q}|}$$

is equivalent to the statement

$$|Z| = \textit{order of} \bigcup_{\alpha} K_\alpha.$$

Proof. Let Z^* be the dual group of Z, i.e., $Z^* = \mathrm{End}_Z(Z, \mathbf{C}^*)$. It is well-known that $Z^* \approx Z$. Consider $K_\alpha \in S$, and define $K_\alpha^\perp = \{\varphi \in Z^* | \varphi(K_\alpha) = 1\}$. Clearly $(Z/K_\alpha)^* \approx K_\alpha^\perp$. Then

$$\frac{|Z|}{|K_\alpha|} = |K_\alpha^\perp|. \tag{4.4}$$

If $\eta\colon Z^* \to Z$ is an isomorphism, then for $K_\alpha \in S$ let $\hat{K}_\alpha = \eta(K_\alpha^\perp \mathrm{k})$. Hence

$$\frac{|Z|}{|K_\alpha|} = |\hat{K}_\alpha|. \tag{4.5}$$

Also for $K_{\alpha_1}, K_{\alpha_2} \in S$ we have

$$\bar{K}_{\alpha_1\alpha_2} = \bar{K}_{\alpha_1} \cap \bar{K}_{\alpha_2}, \tag{4.6}$$

since

$$\hat{K}_{\alpha_1\alpha_2} = \eta(K_{\alpha_1\alpha_2}^\perp) = \eta(K_{\alpha_1}^\perp \cap K_{\alpha_2}^\perp) = \eta(K_{\alpha_1}^\perp) \cap \eta(K_{\alpha_2}^\perp).$$

Hence the sum is

$$\sum_{\alpha}|\hat{K}_\alpha| - \sum_{\alpha_1<\alpha_2}|\hat{K}_{\alpha_1}\cap\hat{K}_{\alpha_2}| + \cdots = |\bigcup_{\alpha} K_\alpha| \tag{4.7}$$

as claimed.

Proof of Theorem 8. Assume the contrary that all but finitely many eigenfunctions are invariant under some nontrivial subgroup of $Z(G)$. Let $S = \{K_\alpha\}_{\alpha=1}^{q}$ be the set of proper subgroups of $Z(G)$. This is a finite set because of Lemma 10. To each $\lambda_i \in \mathrm{Spec}(M)$, we associate an eigenfunction φ_i with eigenvalue λ_i such that the set $\{\varphi_i\}_{i=1}^{\infty}$ form an orthonormal basis for $L^2(M)$, where the λ_i are ordered as follows $0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots$ (including multiplicities). We denote n^λ to be the number of eigenfunctions in $\{\varphi_i\}$ with eigenvalues less than or equal to λ, and n_α^λ (respectively, n_0^λ) be the number of such eigenfunctions which are (respectively, are not invariant under the group K_α. A simple counting argument shows)

$$n^\lambda - n_0^\lambda = \sum_{\alpha} n_\alpha^\lambda - \sum_{\alpha_1<\alpha_2} n_{\alpha_1\alpha_2}^\lambda + \sum_{\alpha_1<\alpha_2<\alpha_3} n_{\alpha_1\alpha_2\alpha_3}^\lambda - \cdots \pm n_{12\cdots q}^\lambda, \tag{4.8}$$

where $n_{\alpha_1\cdots\alpha_p}^\lambda$ = number of eigenfunctions in $\{\varphi_i\}$ with eigenvalues less than or equal to λ and are invariant under the subgroup $K_{\alpha_1\cdots\alpha_p}$ of $Z(G)$ generated by $\bigcup_{i=1}^{p} K_{\alpha_i}$. Let $M_{\alpha_1\cdots\alpha_p} = M/K_{\alpha_1\cdots\alpha_p}$ be the manifold which is covered by M with $K_{\alpha_1\cdots\alpha_p}$ as its group of deck transformations. The eigenfunctions on $M_{\alpha_1\cdots\alpha_p}$ are the $K_{\alpha_1\cdots\alpha_p}$-invariant ones on M. Dividing (4.8) by $\lambda^{m/2}$ yields

$$\frac{n^\lambda}{\lambda^{m/2}} - \frac{n_0^\lambda}{\lambda^{m/2}} = \sum_{\alpha}\frac{n_\alpha^\lambda}{\lambda^{m/2}} - \sum_{\alpha_1<\alpha_2}\frac{n_{\alpha_1\alpha_2}^\lambda}{\lambda^{m/2}} + \cdots \pm \frac{n_{12\cdots q}^\lambda}{\lambda^{m/2}}. \tag{4.9}$$

Taking the limit as $\lambda \to \infty$, the Weyl formula gives

$$C_m V = \sum_{\alpha} C_m V_\alpha - \sum_{\alpha_1<\alpha_2} C_m V_{\alpha_1\alpha_2} + \cdots \pm C_m V_{12\cdots q}, \tag{4.10}$$

where V = volume of M, $V_{\alpha_1\cdots\alpha_p}$ = volume of $M_{\alpha_1\cdots\alpha_p}$, and C_m = constant depending only on m. Here we have used the fact that $\lim_{\lambda\to\infty} n_0^\lambda$ is finite. Since $M \to M_{\alpha_1\cdots\alpha_p}$ is a covering map with the number of sheets equal to $|K_{\alpha_1\cdots\alpha_p}|$,

$$V_{\alpha_1\cdots\alpha_p} = \frac{V}{|K_{\alpha_1\cdots\alpha_p}|}. \tag{4.11}$$

Therefore (4.10) becomes

$$1 = \sum_{\alpha}\frac{1}{|K_\alpha|} - \sum_{\alpha_1<\alpha_2}\frac{1}{|K_{\alpha_1\alpha_2}|} + \cdots \pm \frac{1}{|K_{12\cdots q}|}. \tag{4.12}$$

Multiplying both sides by $|Z| = |Z(G)|$, we have

$$|Z| = \sum_{\alpha} \frac{|Z|}{|K_\alpha|} - \sum_{\alpha_1 < \alpha_2} \frac{|Z|}{|K_{\alpha_1\alpha_2}|} + \cdots \pm \frac{|Z|}{|K_{12\cdots q}|}. \tag{4.13}$$

By Proposition 11, this is equivalent to the fact that the order of $Z(G)$ is equal to the order of the union of all its proper subgroups. But this is true iff $Z(G)$ is not cyclic. Hence this contradicts the assumption.

Remark. In fact, we have shown that

$$n_0^\lambda \sim C_m \Lambda^{m/2}\left[1 - \frac{\text{order of } \bigcup_\alpha K_\alpha}{|Z|}\right].$$

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